Lecture 23

Monday, October 25, 2021 9:31 AM

* Prayer
* Spiritual thought

Ex Solve the ODE $y'' + xy = e^x$ (without the initial conditions).

Write $y = \sum_{n=0}^{\infty} a_n x^n$.

Still get
$$a_{n+2} = \frac{\frac{1}{n!} - a_{n-1}}{(n+1)(n+2)}.$$

We need a_0 and a_1. Note: a_2 is determined from the ODE:

$$\underbrace{y''(0)}_{2a_2} + \underbrace{0\,y(0)}_{0} = \underbrace{e^0}_{1} \longrightarrow a_2 = \frac{1}{2}$$

Set $a_0 = 0$, $a_1 = 1 \rightsquigarrow$ get y_1

Set $a_0 = 1$, $a_1 = 0 \rightsquigarrow$ get y_2

Note: $W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$

Thus, y_1 and y_2 form a fundamental set of solutions.

Second method to deal with

$$y'' + xy = e^x, \quad y(0) = 0, \quad y'(0) = 1.$$

Write $y = \sum_{n=0}^{\infty} a_n x^n$.

If we want to find only the first few coefficients a_n's, then we don't need to equalize coefficients. Instead, we observe that

$$a_n = \frac{y^{(n)}(0)}{n!}.$$

Thus $a_2 = \frac{y''(0)}{2} = \frac{1}{2}$ (from the ODE).

$$a_3 = \frac{y'''(0)}{6}$$

$$\frac{d}{dx}\left(y'' + xy = e^x\right) \rightsquigarrow y''' + xy' + y = e^x \quad (*)$$

Plug $x = 0$: $\quad y'''(0) + 0 + \underbrace{y(0)}_{0} = \underbrace{e^0}_{1} \rightsquigarrow y'''(0) = 1 \rightsquigarrow a_3 = \frac{1}{6}$

$$a_4 = \frac{y^{(4)}(0)}{24}.$$

Differentiating $(*)$:

$$y^{(4)} + xy'' + y' + y' = e^x$$

Plug $x = 0$:

$$24a_4 + 0 + \underbrace{2y'(0)}_{2} = \underbrace{e^0}_{1} \rightarrow a_4 = -\frac{1}{24}.$$

* Limitation of the power series method:

The solution is found as a series. Not all functions can be represented as a series around a point. For example, $\frac{1}{x}$ can't be represented as a power series about $x_0 = 0$. But it can be represented as a power series about $x_0 = 1$.

$$\frac{1}{x} = \frac{1}{(x-1)+1} = \frac{1}{1-[\underbrace{-(x-1)}_{t}]} = 1 + t + t^2 + t^3 + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$